3rd Annual Lexington Mathematical Tournament
Theme Round

Solutions

1 Dragons

1. Answer: $\boxed{1570}$

   Solution: Dragonite flies a total of $2\pi(4000) = 25120$ miles in 16 hours. That is $\frac{25120}{16} = 1570$ miles per hour.

2. Answer: $\boxed{105}$

   Solution: The number of ways to move from (x, y) to $(x + h, y + k)$ by only moving east and north one unit at a time is $\binom{h+k}{k}$, because out of the $(h+k)$ moves, we must choose h of them to be east. From $(0, 0)$ to $(2, 5)$, we have $\binom{7}{2}$ possible moves, and from $(2, 5)$ to $(6, 6)$, we have $\binom{5}{1}$ possible moves. Thus, there are $\binom{7}{2} \cdot \binom{5}{1} = 105$ possible paths that Toothless can take.

3. Answer: $\boxed{22}$

   Solution: On the kth day, Smaug steals $1 + 2 + \cdots + k = \frac{k(k+1)}{2} = \binom{k+1}{2}$ pieces of treasure. Therefore, he will have stolen $\binom{2}{2} + \binom{3}{2} + \cdots + \binom{k+1}{2}$ pieces of trasure by the kth day. By the hockey-stick theorem, this expression is simply $\binom{k+2}{3}$. So we must find the smallest integer k for which $\binom{k+2}{3} \geq 2012$. By trial and error, we find that the answer is 22.

4. Answer: $\boxed{120}$

   Solution: Temeraire can read the math books in one of four ways: (1) first, third, and fifth; (2) first, third, and sixth; (3) first, fourth, and sixth; (4) second, fourth, and sixth. These can be split up into two cases:
   Case 1 and 4: There are no restrictions as to when to read the other three books. There are 3! ways to order the math books and 3! ways to order the others, for a total of 36 ways.
   Case 2 and 3: One of the science books must be read between the two math books. There are 3! ways to order the math books, 2 ways to choose the science book, and 2! ways to order the rest, for a total of 24 ways.
   Thus, there are a total of $2 \cdot 36 + 2 \cdot 24 = 120$ ways in which Temeraire can read the six books.

5. Answer: $\boxed{\dfrac{200}{3}}$

   Solution: Assume that Saphira flies at r miles per hour. After t hours, Thorn is $50 - 50t$ miles east and rt miles south of Saphira. From the Pythagorean theorem, we get that he is $\sqrt{(50 - 50t)^2 + (rt)^2} = \sqrt{(2500 + r^2)t^2 - 5000t + 2500}$ miles away from her. This value must be greater than or equal to 40, so $(2500 + r^2)t^2 - 5000t + 2500 \geq 1600$. The minimum value of a quadratic $y = ax^2 + bx + c$ is achieved when $x = -\frac{b}{2a}$ and $y = c - \frac{b^2}{4a}$. Therefore, $c - \frac{b^2}{4a} = 2500 - \frac{(-5000)^2}{4(2500+r^2)} \geq 1600 \Rightarrow \frac{5000^2}{4(2500+r^2)} \leq 900 \Rightarrow 2500 + r^2 \geq \frac{62500}{9} \Rightarrow r \geq \frac{200}{3}$. Thus, Saphira must fly at a rate of at least $\frac{200}{3}$ miles per hour.

2 Knights and Knaves

6. Answer: $\boxed{16}$

   Solution: There are $\binom{5}{3} = 10$ to choose three knights, $\binom{5}{4} = 5$ ways to choose four knights, and $\binom{5}{5} = 1$ way to choose five. Therefore, there are $10 + 5 + 1 = 16$ total possible ways.

7. Answer: $\boxed{\text{Ali and Cam}}$

   Solution: First, assume that Ali tells the truth. Then, either Bob and Dan must be a knave. If Bob is a knave, then Ali and Eve must both be knaves. Since we have already assumed that Ali is a knight, we have a contradiction. Now, assume that Dan is a knave. Then, Ali and Cam are both knights. We know that Bob must be a knight, so Eve must be lying. However, we have exactly two knaves this way, so we again arrive at a contradiction. Therefore, Ali must be lying. Both Bob and Dan are knights. From what Bob says, Eve must be a knight. From Dan, Cam must be lying. Everything works out, so we have Ali and Cam as knaves.

8. Answer: $\boxed{16}$

   Solution: For each pair of two players sitting opposite to each other, they can either both be knights or both be knaves. Since there are 4 such pairs of players, there are $2^4 = 16$ ways in total to assign the roles.

9. Answer: $\boxed{4}$

   Solution: For each knight, there must be at most four knaves, one on each side of the knight. Therefore, we must have at least 4 knights. 4 knights can be achieved by placing them on the sides so that each knight is separated from two other knights by one knight's move in chess. (Coincidence?)

10. Answer: $\boxed{43}$

    Solution: Case 1: The player in the top-left corner is a knight. Then, that player's two adjacent players must be knaves. Then, the leftmost unassigned player must be a knight. It turns out that for each unassigned column of players, save the rightmost column, we must assign one player to be a knight and one to be a knave. However, we cannot have three of the same arrangements in a row, or else the knave in the middle column would be correct. The rightmost column must be arranged in the opposite way as the one to its left. For $0 \le x \le 7$, let $f(x) = 0$ if the column to the right of the $(x+1)$th column, counting from the left, is arranged in the same way as the xth column, and let $f(x) = 1$ otherwise. We know that $f(0) = f(7) = 1$, so we seek the number of ways to assign the rest. Our previous restriction tells us that for $0 \le x \le 6$, we must not have $f(x) = f(x+1) = 0$.
    Let $g(n)$ denote the number of ways to assign $f(x)$ for $1 \le x \le n$. Then, we look for $g(6)$. Assume at first that the domain of $f(x)$ is $x = 1$. Then, $g(1) = 2$. For $1 \le x \le 2$, we can see that $g(2) = 3$. If we let $1 \le x \le k$ for some larger k, we see that if we let $f(k) = 1$, we have $g(k-1)$ ways to assign the rest without restriction, but if we let $f(k) = 0$, $f(k-1)$ must be 1 and we have $g(k-2)$ ways to assign the rest without restriction. Therefore, $g(k) = g(k-1) + g(k-2)$ for $k > 2$. Using this recursive formula, we get that $g(3) = 5$, $g(4) = 8$, $g(5) = 13$, and $g(6) = 21$. Thus, there are 21 total possibilities in this case.
    Case 2: The player in the top-left corner is a knave. If we let the left-most unassigned player be a knight, then we have the same argument as in Case 1, which results to 21 ways. If we let that player be a knave, then the two players in the left-most unassigned column must be knights. Each subsequent column must either be composed of two knights or two knaves and be different from its adjacent columns. There is 1 way to do this.
    In total, there are $21 + 21 + 1 = 43$ possible ways.

3 Evaluate-athon

11. Answer: $\boxed{351}$

    Solution: The kth term of this arithmetic series is $3+4(k-1)$. The last term is the 13th term, which we get by solving $3 + 4(k-1) = 51$. The total sum of the series $a_1 + a_2 + \cdots + a_n$ is therefore $\frac{n}{2}(a_1 + a_n) = \frac{13}{2}(3 + 51) = 351$.

12. Answer: $\boxed{555555}$

Solution: We see that we can rewrite $123 \cdot 357 + 123 \cdot 644 + 432 \cdot 357 + 432 \cdot 644$ as $123(357 + 644) + 432(357 + 644) = (123 + 432)(357 + 644) = 555 \cdot 1001 = 555555$.

13. Answer: $\boxed{4052171}$

Solution: We let $2012 = 2011 + 1$ and compute as follows: $\lfloor \frac{(2011+1)^4}{2011^2} \rfloor = \lfloor (\frac{(2011+1)^2}{2011})^2 \rfloor = \lfloor (\frac{2011^2+2\cdot 2011+1}{2011})^2 \rfloor = \lfloor (2011 + 2 + \frac{1}{2011})^2 \rfloor = \lfloor (2013 + \frac{1}{2011})^2 \rfloor = \lfloor 2013^2 + 2 \cdot \frac{2013}{2011} + \frac{1}{2011^2} \rfloor = \lfloor 4052169 + 2 + \frac{4}{2011} + \frac{1}{2011^2} \rfloor = \lfloor 4052171 + \frac{4}{2011} + \frac{1}{2011^2} \rfloor = 4052171$.

14. Answer: $\boxed{\dfrac{29525}{1024}}$

Solution: $\binom{10}{10} + \binom{10}{8}(\frac{1}{2})^2 + \binom{10}{6}(\frac{1}{2})^4 + \cdots + \binom{10}{0}(\frac{1}{2})^{10} = \frac{1}{2}([\binom{10}{10} + \binom{10}{9}(\frac{1}{2})^1 + \binom{10}{8}(\frac{1}{2})^2 + \binom{10}{7}(\frac{1}{2})^3 + \ldots + \binom{10}{1}(\frac{1}{2})^9 + \binom{10}{0}(\frac{1}{2})^{10}] + [\binom{10}{10} - \binom{10}{9}(\frac{1}{2})^1 + \binom{10}{8}(\frac{1}{2})^2 - \binom{10}{7}(\frac{1}{2})^3 + \ldots - \binom{10}{1}(\frac{1}{2})^9 + \binom{10}{0}(\frac{1}{2})^{10}])$. By the binomial theorem, this expression becomes $\frac{1}{2}((1+\frac{1}{2})^{10} + (1-\frac{1}{2})^{10}) = \frac{1}{2}(\frac{3^{10}+1}{2^{10}}) = \frac{29525}{1024}$.

15. Answer: $\boxed{\dfrac{7}{8}}$

Solution: We observe and claim that $(1^{-3} + 2^{-3} + (2^2)^{-3} + (2^3)^{-3} + \cdots)(1^{-3} + 3^{-3} + 5^{-3} + 7^{-3} + \cdots) = 1^{-3} + 2^{-3} + 3^{-3} + 4^{-3} + \cdots$. We can see that this equation holds by showing that any k^{-3}, $k > 0$, can be expressed uniquely as the product of one term from each infinite series on the left hand side.
Let $k = 2^m \times n$ such that $(2^m)^{-3}$ is a term in $1^{-3} + 2^{-3} + (2^2)^{-3} + (2^3)^{-3} + \cdots$ and n^{-3} is a term in $1^{-3} + 3^{-3} + 5^{-3} + 7^{-3} + \cdots$. n must be odd, and one way to satisfy this condition is to let m equal the number of powers of 2 that n has. Now, if m is any higher, then n is not an integer, and if m is any lower, n becomes even. Thus, m and n are unique.
$1^{-3} + 2^{-3} + (2^2)^{-3} + (2^3)^{-3} + \cdots$ is an infinite geometric series that sums to $\frac{1}{1-2^{-3}} = \frac{8}{7}$. Thus, $\frac{1^{-3}+3^{-3}+5^{-3}+7^{-3}+\cdots}{1^{-3}+2^{-3}+3^{-3}+4^{-3}+\cdots} = \frac{1^{-3}+3^{-3}+5^{-3}+7^{-3}+\cdots}{\frac{8}{7}(1^{-3}+3^{-3}+5^{-3}+7^{-3}+\cdots)} = \frac{7}{8}$.
Additional note: Nobody actually knows the exact values of $1^{-3} + 2^{-3} + 3^{-3} + 4^{-3} + \cdots$ and $1^{-3} + 3^{-3} + 5^{-3} + 7^{-3} + \cdots$! However, the ratio of the two expressions can be found without knowing the values of the expressions themselves.